ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 244 (2008) 1080–1092

Journal of Differential Equations

www.elsevier.com/locate/jde

Analytic normalization of analytic integrable systems and the embedding flows

Xiang Zhang [1]

Department of Mathematics, Shanghai Jiaotong University, Shanghai 200240, People's Republic of China

Received 8 April 2007; revised 8 November 2007

Abstract

This paper provides the normal forms of analytic integrable differential systems and diffeomorphisms via analytic normalizations. Furthermore, we consider the existence of embedding flows of an analytic integrable diffeomorphism.

MSC: 34A25; 34A34; 34C20; 37C15

Keywords: Analytic system; Analytic integrable; Normal form; Normalization; Embedding flow

1. Introduction and statement of the main results

The problem on if an analytic system is analytically equivalent to its normal form is classical. It is well known [8,9,13,17] that the existence of analytic normalizations transforming an analytic vector field to a desired normal form is strongly related to the existence of analytic first integrals.

For general planar analytic systems with a singularity degenerate or non-degenerate, Llibre et al. [2,3,6] characterize their local analytic integrability with the aid of normal forms. From the classical Poincaré theorem [13] we have that for a planar analytic system, the origin is a non-degenerate analytic center of the system if and only if it is analytically equivalent to

$$\dot{x} = x\big(1 + q(xy)\big), \qquad \dot{y} = -y\big(1 + q(xy)\big), \tag{1.1}$$

E-mail address: xzhang@sjtu.edu.cn.

[1] The author is partially supported by the NNSF of China grant No. 10671123 and NCET of China grant No. 050391.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2008.01.001

via the transformation of complex variables and of the time probably, where $q(u)$ is an analytic function in u starting from the terms of degree no less than 1. For planar analytic Hamiltonian systems, if the origin is a hyperbolic saddle, then there exists a real analytic area-preserving transformation of the variables for which the system is changed to Eq. (1.1) (see e.g. [11]). These results were extended to general Hamiltonian systems by Ito [8,9]. He proved that an analytic Liouvillian integrable Hamiltonian system with the eigenvalues non-resonant or only one resonant at a singularity is analytically symplectically equivalent to its Birkhoff normal form. Recently, Zung [17] completely solved the problem, i.e. without any restriction on the resonance. In other words, any analytically Liouvillian integrable Hamiltonian system is analytically symplectically equivalent to its Birkhoff normal form. Siegel [14] proved that if the symplectic transformation reducing an analytic Hamiltonian H to its Birkhoff normal form is convergent, then the Hamiltonian system has exactly n functionally independent analytic first integrals. Furthermore, he proved that in the set of Hamiltonians having the same second order terms as that of H, then there exists a dense subset endowed with the coefficient topology, in which every Hamiltonian vector field has only itself as the functionally independent analytic first integral. Consequently, it cannot be reduced to its Birkhoff normal form by an analytic symplectic transformation.

For general analytic differential systems in n-dimensional Euclidean spaces, it is an open problem that *if an analytically integrable system is analytically equivalent to its normal form*. In this short note, we will solve this problem in the case of non-degeneracy. Also we will consider the existence of the analytic normalization and the embedding flow of an analytic integrable diffeomorphism. Recall that an n-dimensional analytic differential system or vector field is *analytically integrable* if it has $n-1$ functionally independent analytic first integrals. *An analytic first integral* of an analytic vector field $\mathcal{X}$ is an analytic function and is a constant along each orbit of $\mathcal{X}$.

Consider the following analytic system

$$\dot{x} = Ax + F(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $F(x) = O(|x|^2)$ is an analytic vector-valued function in $(\mathbb{R}^n, 0)$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the n-tuple of eigenvalues of the matrix A. Set

$$\mathcal{M}_\lambda := \left\{ m = (m_1, \ldots, m_n);\ \langle m, \lambda \rangle = \sum_{i=1}^{n} m_i \lambda_i = 0,\ m_i \in \mathbb{Z}_+,\ |m| \geqslant 1 \right\},$$

where $\mathbb{Z}_+$ denotes the set of non-negative integers, and $|m| = m_1 + \cdots + m_n$. Denote by R_λ the rank of vectors in the set $\mathcal{M}_\lambda$. Then $R_\lambda \leqslant n-1$. The following is our main results.

Theorem 1.1. *Assume that the origin of system* (1.2) *is non-degenerate, i.e. no eigenvalues equal to zero, and that the matrix A can be diagonalizable. Then system* (1.2) *has* $n-1$ *locally functionally independent analytic first integrals if and only if* $R_\lambda = n-1$, *and system* (1.2) *is analytically equivalent to its distinguished normal form*

$$\dot{y}_i = \lambda_i y_i \big(1 + g(y)\big), \quad i = 1, \ldots, n, \tag{1.3}$$

by an analytic normalization, where $g(y)$, *without constant terms, is an analytic function of* y^m *with* $m \in \mathcal{M}_\lambda$ *and* $(m_1, \ldots, m_n) = 1$, *i.e. there is no common factor.*

Remark 1. Our results generalize the classical Poincaré theorem mentioned above, and the Ito's one on Hamiltonian systems with degree 1 of freedom (also the one in [11]). In fact, these results are the special cases of our results.

Remark 2. If the origin of system (1.2) is degenerate, i.e. there are zero eigenvalues, the problem is still open. That is to say, if system (1.2) has the origin as a degenerate singularity, and has $n-1$ locally functionally independent analytic first integrals in a neighborhood of the origin, is system (1.2) locally analytically equivalent to its distinguished normal forms?

Similar to vector fields, a diffeomorphism $F(x)$ defined on an analytic manifold $\mathcal{M}$ is *analytic integrable* if it has $n-1$ functionally independent analytic first integrals. An *analytic first integral* of $F(x)$ is an analytic function $V(x)$ which satisfies $V(F(x)) = V(x)$ for all $x \in \mathcal{M}$.

Theorem 1.2. *If $F(x) = Bx + f(x)$ is an n-dimensional locally analytic integrable diffeomorphism with B diagonalizable, then it is analytically equivalent to the diffeomorphism* $\mathrm{diag}(\mu_1 y_1, \ldots, \mu_n y_n)(1 + h(y))$*, where μ_i are the eigenvalues of B and $h(y)$ is an analytic function containing only resonant terms and with no constant term.*

In higher dimensions, on the embedding of diffeomorphisms in flows, Palis [12] proved that the diffeomorphisms that can be embedded in flows are rare in the Baire sense. In [10], we provided some sufficient conditions for a given C^∞ diffeomorphism to admit an embedding flow. In the analytic world, the embedding problem that can be solved is related only to the diffeomorphisms which can be analytically linearized [15], or those whose linear part has the eigenvalues belonging to the Poincaré domain [16]. For integrable diffeomorphisms we have the following.

Theorem 1.3. *Any analytic integrable volume-preserving diffeomorphism defined on an analytic manifold $\mathcal{M}$ can be embedded in an analytic flow on $\mathcal{M}$.*

Recall that a diffeomorphism $F(x)$ defined on a smooth manifold $\mathcal{M}$ can be embedded in a vector field $\mathcal{X}$ on $\mathcal{M}$ if it is the time 1 map of the flow induced by $\mathcal{X}$, i.e. $\mathcal{X} \circ F(x) = DF(x)\mathcal{X}(x)$ for all $x \in \mathcal{M}$. We also say that the diffeomorphism can be embedded in the flow. Cima et al. [5] investigated some relation of the dynamics between diffeomorphisms and its embedding flows.

The paper is organized as follows. In Section 2 we give the proof of Theorem 1.1. In Section 3 we prove Theorems 1.2 and 1.3.

2. Proof of Theorem 1.1

Sufficiency. Since $R_\lambda = n-1$, there exist $m_i = (m_{i1}, \ldots, m_{in}) \in \mathbb{Z}_+^n$, $i = 1, \ldots, n$, such that the $n-1$ vectors are independent and satisfy $\langle m_i, \lambda\rangle = 0$. Some simple calculations show that y^{m_i}, $i = 1, \ldots, n-1$, are $n-1$ functionally independent first integrals of (1.3). If $x = y + \Phi(y)$ is the analytic transformation reducing (1.2) to (1.3) in a neighborhood of the origin with the inverse $y = \chi(x)$, then χ^{m_i}, $i = 1, \ldots, n-1$, are the $n-1$ functionally independent analytic first integrals of (1.2).

Necessity. Denote by $\mathcal{X}$ the vector fields induced by system (1.2). Set $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_h$, where $\mathcal{X}_1$ and $\mathcal{X}_h$ are the linear and higher order terms, respectively. Since the algebra of linear vector fields in $\mathbb{R}^n$, under the standard Lie bracket, is nothing but the reductive algebra

$gl(n,\mathbb{R}) = sl(n,\mathbb{R}) \oplus \mathbb{R}$, we write $A = A_1 + A_2$ with A_1 semisimple and A_2 nilpotent. Correspondingly, we separate $\mathcal{X}_1 = \mathcal{X}_1^s + \mathcal{X}_1^n$ with $\mathcal{X}_1^s = \langle A_1 x, \partial_x \rangle$ called the *semisimple part* of $\mathcal{X}_1$ and $\mathcal{X}_1^n = \langle A_2 x, \partial_x \rangle$ called the *nilpotent part* of $\mathcal{X}_1$. Without loss of generality, we can assume that

$$\mathcal{X}_1^s := \sum_{i=1}^{n} \lambda_i x_i \frac{\partial}{\partial x_i}.$$

We say that the vector field $\mathcal{X}$ is in *normal form* if the Lie bracket of $\mathcal{X}_1^s$ and $\mathcal{X}_h$ vanishes, i.e. $[\mathcal{X}_1^s, \mathcal{X}_h] = 0$. We note that system (1.2) is in normal form means that all monomials of system (1.2) are resonant. Recall that a monomial x^m in the sth component of system (1.2) is *resonant* if $\langle m, \lambda \rangle = \lambda_s$. A monomial x^m in a function is *resonant* if $\langle m, \lambda \rangle = 0$.

For a given analytic system or vector field, by the Poincaré–Dulac normal form theorem it can always be transformed to a normal form by a formal transformation. But usually, a transformation reducing a vector field to its normal form is not unique. In what follows, we call such a transformation *distinguished normalization* if it contains non-resonant terms only. The distinguished normalization is unique. Correspondingly, the normal form is called a *distinguished normal form*.

The following result, due to Bibikov [1], will be used in the proof of the existence of normal forms.

Lemma 2.1. *Denote by $\mathcal{G}^r(\mathbb{R})$ the linear space of n-dimensional vector-valued homogeneous polynomials of degree r in n variables with coefficients in $\mathbb{R}$. Let A and B be two nth square matrices with entries in $\mathbb{R}$, and their n-tuples of eigenvalues be λ and κ, respectively. Define a linear operator L on $\mathcal{G}^r(\mathbb{R})$ as follows,*

$$Lh = \langle \partial_x h, Ax \rangle - Bh, \quad h \in \mathcal{G}^r(\mathbb{R}).$$

Then the spectrum of the operator L is

$$\sigma(L) := \left\{ \langle l, \lambda \rangle - \kappa_j;\ l \in \mathbb{Z}_+^n,\ |l| = r,\ j = 1, \ldots, n \right\}.$$

The proof of Theorem 1.1 follows from the following lemmas. The first one shows the existence of the distinguished normal form of a given analytic system. Its proof is not completely new, but we will use the proof in the following, so we present it here.

Lemma 2.2. *System* (1.2) *can be transformed to its distinguished normal form by a distinguished normalization.*

Proof. Assume that system (1.2) is transformed to

$$\dot{y} = Ay + G(y), \tag{2.1}$$

by a diffeomorphism (analytically or formally)

$$x = y + \Phi(y), \tag{2.2}$$

where $G(y)$ and $\Phi(y)$ are series (analytically or formally) starting from the terms of degree no less than 2. Write

$$W(z) = \sum_{s=2}^{\infty} W_s(z),$$

with $W \in \{F, G, \Phi\}$, where $W_s(z)$ is a homogeneous vector-valued polynomial of degree s. Then G_s and Φ_s satisfy the following

$$\langle \partial_y \Phi_s, Ay\rangle - A\Phi_s = [F]^s - \sum_{j=2}^{s-1} \partial_y \Phi_j(y) G_{s+1-j}(y) - G_s(y), \tag{2.3}$$

where $[F]^s$ are homogeneous vector-valued polynomials obtained after we re-expand $F(y + \Phi(y))$ in power series in y.

Set

$$L := \langle \partial_y, Ay\rangle - A.$$

By Lemma 2.1 we have on the linear space $\mathcal{G}^s(\mathbb{R})$

$$\sigma(L) = \left\{\langle m, \lambda\rangle - \lambda_j;\ \lambda = (\lambda_1, \ldots, \lambda_n),\ \lambda_j \in \sigma(A),\ m \in \mathbb{Z}_+^n,\ |m| = s\right\}.$$

Separate $\mathcal{G}^s(\mathbb{R}) = \mathcal{G}_0^s(\mathbb{R}) \oplus \mathcal{G}_1^s(\mathbb{R})$ such that $L = 0$ acting on the former, and L is invertible on the later.

For obtaining the distinguished normal forms, we separate the components in the right-hand side of (2.3) into two parts according to the decomposition $\mathcal{G}_0^s(\mathbb{R}) \oplus \mathcal{G}_1^s(\mathbb{R})$. For the part belonging to $\mathcal{G}_0^s(\mathbb{R})$, we choose $\Phi_s(y) = 0$ and

$$G_s(y) = [F]^s - \sum_{j=2}^{s-1} \partial_y \Phi_j(y) G_{s+1-j}(y).$$

For the part belonging to $\mathcal{G}_1^s(\mathbb{R})$, since L is invertible on this subspace, we choose $G_s(y) = 0$ and $\Phi_s(y)$ is the corresponding unique solution of (2.3).

From the above construction, we get the distinguished normalization (recall that it is the transformation containing non-resonant terms only) reducing system (1.2) to its distinguished normal form (2.1). □

By the assumption of the main theorem, without loss of generality we can suppose in the following that $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$.

Lemma 2.3. *Assume that $H(x)$ is an analytic first integral of* (1.2)*, and that* (2.1) *is the distinguished normal form of* (1.2) *via the distinguished normalization* (2.2)*. Then $\widetilde{H}(y) = H(y + \Phi(y))$ is an analytic first integral of* (2.1)*, and it contains resonant terms only.*

Proof. Since $H(x)$ is an analytic first integral of (1.2), it should satisfies the following

$$\langle \partial_x H, Ax + F(x) \rangle = 0.$$

By the chain rule, we get that

$$\langle \partial_y \widetilde{H}, (\partial_y \widetilde{\Phi})^{-1}(A + F) \circ \widetilde{\Phi}(y) \rangle = 0,$$

where $\widetilde{\Phi}(y) = y + \Phi(y)$ is the change of variables given in (2.2). This means that $\widetilde{H}(y)$ is a first integral (analytically or formally) of system (2.1).

Write

$$\widetilde{H}(y) = \sum_{k=l}^{\infty} \widetilde{H}_k(y),$$

with $l \geqslant 1$. Since $\widetilde{H}(y)$ is a first integral of (2.1), then we have

$$\sum_{i=1}^{n} \lambda_i y_i \partial_{y_i} \widetilde{H}_l = 0, \quad \text{i.e.} \quad \langle \partial_y \widetilde{H}_l, Ay \rangle = 0.$$

So, $\widetilde{H}_l$ contains resonant terms only.

We assume that for any given $m > l$, $\widetilde{H}_j$, $j = l, \dots, m-1$, are all resonant. Simple computations show that

$$\langle \partial_y \widetilde{H}_m, Ay \rangle + \sum_{j=2}^{m} \langle \partial_y \widetilde{H}_{m+1-j}, G_j \rangle = 0.$$

Since G_j and $\widetilde{H}_{m+1-j}$ are resonant homogeneous polynomials in a vector field and in a function, respectively, the second component in the above summation contains resonant terms only as a function. Consequently, so is the first component. Therefore, $\widetilde{H}_m$ is resonant.

By the induction, we have proved that $\widetilde{H}$ contains resonant terms only. □

The following gives the distinguished normal form of an analytic integrable system.

Lemma 2.4. *If system* (1.2) *has* $n-1$ *functionally independent analytic first integrals, then its distinguished normal form has the following form*

$$\dot{y}_i = \lambda_i y_i \big(1 + g(y)\big), \quad i = 1, \dots, n, \tag{2.4}$$

where $g(y)$ *is a series starting from the term of degree at least one.*

Proof. Denote by $\widetilde{\mathcal{X}} := (\lambda_1 y_1 + g_1(y), \dots, \lambda_n y_n + g_n(y))$ the distinguished vector fields defined by (2.1). Suppose that $H_1(x), \dots, H_{n-1}(x)$ are the $n-1$ functionally independent first integrals of (1.2). From Lemma 2.3, the vector field $\widetilde{\mathcal{X}}$ has $n-1$ first integrals $\widetilde{H}_1(y), \dots, \widetilde{H}_{n-1}(y)$, which are functionally independent and all resonant.

Denote by Ω the linear space formed by $\{\partial_y \widetilde{H}_i,\ i = 1, \dots, n-1\}$. Since $\widetilde{H}_i(y)$, $i = 1, \dots, n-1$, are first integrals of the vector field $\widetilde{\mathcal{X}}$, it follows from the definition of first integrals that the vector field $\widetilde{\mathcal{X}}$ is orthogonal to the $(n-1)$-dimensional linear space Ω. All the first integrals $\widetilde{H}_i(y)$ are resonant, it implies that $\langle \partial_y \widetilde{H}_i(y), Ay\rangle = 0$. This proves that $(\lambda_1 y_1, \dots, \lambda_n y_n)$ is also orthogonal to Ω. Since we are in the n-dimensional space, the two vector fields $\widetilde{\mathcal{X}}$ and $(\lambda_1 y_1, \dots, \lambda_n y_n)$ should be parallel at each point y in a neighborhood of the origin. Hence, there exists a function of the form $1 + g(y)$ such that $\widetilde{\mathcal{X}} = (\lambda_1 y_1(1+g(y)), \dots, \lambda_n y_n(1+g(y)))$. $\square$

Lemma 2.5. *If system* (1.2) *has $n-1$ analytic first integrals, then there exists $\kappa > 0$ such that for all $\langle m, \lambda\rangle - \lambda_i \neq 0$, $m \in \mathbb{Z}_+^n$, $|m| \geqslant 2$, we have*

$$\left|\langle m, \lambda\rangle - \lambda_i\right| > \kappa.$$

Proof. By Theorem 1.1 of [4], i.e. the number of analytic first integrals of system (1.2) is less than or equal to R_λ, and $R_\lambda \leqslant n-1$, under the assumption of the lemma the n-tuple λ of eigenvalues of the matrix A should satisfy $n-1$ resonant relations:

$$\begin{aligned} m_{1,1}\lambda_1 + \cdots + m_{1,n}\lambda_n &= 0, \\ \vdots \qquad\qquad & \vdots \\ m_{n-1,1}\lambda_1 + \cdots + m_{n-1,n}\lambda_n &= 0, \end{aligned} \tag{2.5}$$

where the $n-1$ vectors $(m_{1,1}, \dots, m_{1,n}), \dots, (m_{n-1,1}, \dots, m_{n-1,n}) \in \mathbb{Z}_+^n$ are linearly independent.

Without loss of generality, we can assume that

$$\det \begin{pmatrix} m_{1,1} & \dots & m_{1,n-1} \\ \vdots & & \vdots \\ m_{n-1,1} & \dots & m_{n-1,n-1} \end{pmatrix} \neq 0.$$

Then solving (2.5) yields

$$\lambda_1 = \frac{\nu_1}{\mu_1}\lambda_n, \quad \dots, \quad \lambda_{n-1} = \frac{\nu_{n-1}}{\mu_{n-1}}\lambda_n,$$

with $\mu_i, \nu_i \in \mathbb{Z} \setminus \{0\}$, and μ_i, ν_i relatively prime for $i = 1, \dots, n-1$.

For $\langle m, \lambda\rangle - \lambda_i \neq 0$, $m \in \mathbb{Z}_+^n$, $|m| \geqslant 2$, the following hold

$$\left|\langle m, \lambda\rangle - \lambda_i\right| = \frac{\Lambda_i}{\mu_1 \cdots \mu_{n-1}}|\lambda_n| \geqslant \frac{|\lambda_n|}{\mu_1 \cdots \mu_{n-1}},$$

where $\Lambda_i \in \mathbb{N}$. Then $\kappa = \min\{|\lambda_i|/(\mu_1 \cdots \mu_{n-1});\ i = 1, \dots, n\}$ is suitable for the lemma. $\square$

Lemma 2.6. *Under the assumption of Theorem* 1.1*, the distinguished normalization* (2.2) *reducing* (1.2) *to* (2.4) *is convergent.*

Proof. For $w(z) \in \{f_i, \phi_i, g\}$ with f_i and ϕ_i are the ith components of F and Φ, respectively, we expand it in

$$w(z) = \sum_{k \in \mathbb{Z}_+^n} w^k z^k,$$

where w^k is the coefficient of the monomial $z^k = z_1^{k_1} \cdots z_n^{k_n}$. From the proof of Lemma 2.2, we get that ϕ_s^k satisfy the following

$$\big(\langle k, \lambda\rangle - \lambda_s\big)\phi_s^k = \big[f_s\big(y + \Phi(y)\big)\big]^k - \lambda_s g^{k-e_s} - \sum_{j=1}^{n} \sum_{l \prec k,\, l \in \mathbb{Z}_+^n} \phi_s^l l_j \lambda_j g^{k-l}, \tag{2.6}$$

where $[f_s]^k := [f_s(y+\Phi(y))]^k$ is the coefficient of y^k obtained after we re-expand $f_s(y+\Phi(y))$ in power series in y, and e_s the n-dimensional unit vector with the sth entry equal to 1, and $l \prec k$ means that $k - l \in \mathbb{Z}_+^n$.

Since we are in the case of the distinguished normalization, if $\langle k, \lambda\rangle - \lambda_s = 0$, by Lemma 2.2, Eq. (2.6) has the solutions

$$\phi_s^k = 0, \qquad g^{k-e_s} = \lambda_s^{-1}\Bigg([f_s]^k - \sum_{l \prec k,\, l \in \mathbb{Z}_+^n} \langle l, \lambda\rangle \phi_s^l g^{k-l}\Bigg). \tag{2.7}$$

If $\langle k, \lambda\rangle - \lambda_s \neq 0$, solving Eq. (2.6) with the choice of $g^{k-e_s} = 0$ yields

$$\phi_s^k = \frac{[f_s]^k - \sum_{l \prec k,\, l \in \mathbb{Z}_+^n} \langle l, \lambda\rangle \phi_s^l g^{k-l}}{\langle k, \lambda\rangle - \lambda_s}. \tag{2.8}$$

We claim that in Eq. (2.7)

$$g^{k-e_s} = \lambda_s^{-1}[f_s]^k.$$

Indeed, by the construction g^{k-l} is the coefficient of resonant terms. Hence, $\langle k - l, \lambda\rangle = 0$. Consequently, we have

$$\langle l, \lambda\rangle = \langle k, \lambda\rangle = \lambda_s.$$

This means that ϕ_s^l is the coefficient of a resonant term. So, it should be equal to zero, because our normalization is distinguished. The claim follows.

Summarizing the above calculations, we achieve the distinguished normalization

$$x_s = y_s + \sum_{k \in \mathbb{Z}_+^n,\, |k|>1} \phi_s^k y^k,$$

with ϕ_s^k satisfying (2.8), and the distinguished normal form

$$\dot{y}_s = \lambda_s y_s \Bigg(1 + \sum_{0 \neq k-e_s \in \mathbb{Z}_+^n} g^{k-e_s} y^{k-e_s}\Bigg), \quad s = 1, \ldots, n,$$

with g^{k-e_s} satisfying (2.7), and $\langle k - e_s, \lambda\rangle = 0$.

To prove the convergence of the normalization, we first estimate ϕ_s^k. From the proof of Lemma 2.5, there exists a positive number δ such that for $|\langle k,\lambda\rangle - \lambda_j| \neq 0$, $k \in \mathbb{Z}_+^n$, $|k| \geqslant 2$, we have

$$\left|\langle k,\lambda\rangle - \lambda_j\right|^{-1} \leqslant \delta.$$

Then

$$\begin{aligned}\left|\frac{\sum_{l\prec k,\, l\in\mathbb{Z}_+^n}\langle l,\lambda\rangle\phi_s^l g^{k-l}}{\langle k,\lambda\rangle-\lambda_s}\right| &\leqslant \sum_{l\prec k,\, l\in\mathbb{Z}_+^n}\left(1+\frac{|\lambda_s|}{|\langle k,\lambda\rangle-\lambda_s|}\right)\left|\phi_s^l g^{k-l}\right| \\ &\leqslant \sum_{l\prec k,\, l\in\mathbb{Z}_+^n}\big(1+\delta|\lambda_s|\big)\left|\phi_s^l g^{k-l}\right|,\end{aligned}$$

where we have used the fact that $\langle l,\lambda\rangle = \langle k,\lambda\rangle = \lambda_s$. Set $\rho = \max\{1+\delta|\lambda_s|,\ s=1,\dots,n\}$. We get that

$$\left|\phi_s^k\right| \leqslant \delta\left|[f_s]^k\right| + \rho\sum_{l\prec k,\, l\in\mathbb{Z}_+^n}\left|\phi_s^l g^{k-l}\right|.$$

The function $F(x) = (f_1,\dots,f_n)$ is analytic in a neighborhood of the origin, there exists a polydisc $\mathcal{D} := \{|x_s| < r,\ s=1,\dots,n\}$ in which the following hold

$$\left|[f_s]^k\right| \leqslant Mr^{-|k|}, \qquad M = \max_s \sup_{\partial\mathcal{D}}\big\{|f_s|\big\},$$

by the Cauchy inequality. Define

$$\hat{f}(x) = M\sum_{|k|=2}^{\infty} r^{-|k|}x^k.$$

This is an analytic function in the interior of $\mathcal{D}$, and is a majorant series of f_s, $s=1,\dots,n$. In the following, we denote by $\hat{w}$ the majorant series of a given series w, and represent it as $w \preccurlyeq \hat{w}$ (see for instance [7]).

Direct computations show that

$$\begin{aligned}\sum_{s=1}^n\phi_s + g &\preccurlyeq \sum_{s=1}^n\hat{\phi}_s + \hat{g} \\ &\preccurlyeq (n\delta+\nu)\hat{f}(y+\hat{\phi}) + (\rho+1)\sum_{s=1}^n\hat{\phi}_s\hat{g},\end{aligned} \tag{2.9}$$

where $\nu = \max_{1\leqslant s\leqslant n}\{|\lambda_s|^{-1}\}$.

Since the coefficients in $\hat{\phi}_s$ and $\hat{g}$ are all positive, the convergence of the series $\sum_{s=1}^n \hat{\phi}_s(y) + \hat{g}(y)$ is equivalent to that in the case $y_1 = \cdots = y_n = u$. Set

$$W(u) = \sum_{s=1}^n \hat{\phi}_s(y) + \hat{g}\bigg|_{y_1=\cdots=y_n=u}.$$

Then $W(u) = V(u)u$ with $V(u)$ a series by the construction of $\hat{\phi}_s$ and $\hat{g}$. It follows from (2.9) that

$$V(u)u \preccurlyeq (n\delta + \nu)u^2 \hat{f}_*\big(1 + V(u)\big) + (\rho + 1)V(u)^2u^2, \tag{2.10}$$

where $\hat{f}_*(1 + V(u)) = \hat{f}(u + \hat{\phi}_1(u, \ldots, u), \ldots, u + \hat{\phi}(u, \ldots, u))/u^2$.

Set

$$\Gamma(u, h) := h - (n\delta + \nu)u\hat{f}_*(1 + h) - (\rho + 1)h^2u. \tag{2.11}$$

Obviously, $\Gamma(u, h)$ is analytic in a neighborhood of the origin, and it satisfies

$$\Gamma(0, 0) = 0, \qquad \partial_h \Gamma|_{(0,0)} = 1.$$

By the Implicit Function Theorem, $\Gamma(u, h) = 0$ has a unique analytic solution, denote $h(u)$, in a neighborhood of the origin. Comparing (2.10) and (2.11), we know that $h(u)$ majorizes $V(u)$. Hence, $V(u)$ is analytic in a neighborhood of the origin, and so is $W(u)$. From the previous discussion, we have proved that $\sum_{s=1}^n \hat{\phi}_s + \hat{g}$ is convergent. Consequently, ϕ_s and g are convergent in a neighborhood of the origin. This proves that system (1.2) is analytically equivalent to its distinguished normal form. □

Combining Lemmas 2.2–2.5 and 2.6 we can complete the proof of the necessary part.

We have finished the proof of Theorem 1.1.

3. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2. Since the proof is similar to that of Theorem 1.1, we only give a sketch of proof.

Assume that $F(x) = Bx + f(x)$ is conjugated to $G(y) = By + g(y)$ via a diffeomorphism $x = \Phi(y) = y + \phi(y)$. Then we have

$$\phi(Bx) - B\phi(x) = f\big(x + \phi(x)\big) + \phi(Bx) - \phi\big(Bx + g(x)\big) - g(x). \tag{3.1}$$

Expand f, g, ϕ in Taylor series, and set

$$\mu(x) = \sum_{i=1}^{\infty} \mu_i(x),$$

with $\mu \in \{f, g, \phi\}$, where μ_i is a vector-valued homogeneous polynomial of degree i.

We note that the linear operator on the linear space formed by vector-valued homogeneous polynomials of degree m

$$\mathcal{L}\phi_i(x) = \phi_i(Bx) - B\phi_i(x),$$

has the spectrum

$$\left\{\prod_{i=1}^{n} \mu_i^{m_i} - \mu_j;\ m_i \in \mathbb{Z}_+,\ \sum_{i=1}^{n} m_i = m,\ j = 1, \ldots, n\right\},$$

where $(\mu_1, \ldots, \mu_n) = \mu$ is the n-tuple of eigenvalues of B. Using the standard method as in Lemma 2.2, we can choose the normalization $\Phi(y)$ in which the nonlinear part contains only non-resonant terms, i.e. all monomials $a_m y^m$ in the ith component satisfying $\mu^m - \mu_i \neq 0$. The nonlinear part in $G(y)$ contains only resonant terms.

Since F and G are conjugate, i.e. $F \circ \Phi = \Phi \circ G$, if $V(x)$ is a first integral of $F(x)$, then $V_G(y) = V \circ \Phi(y)$ is a first integral of $G(y)$. Moreover, if G is in the distinguished normal form, then V_G has the nonlinear terms all resonant, i.e. its monomial $v_m y^m$ satisfying $\mu^m = 1$. This implies that V_G is also a first integral of By, where we have supposed without loss that B is a diagonal matrix.

Assume that $V_1, \ldots, V_{n-1}$ are the $n-1$ functionally independent first integrals. Then $V_{iG}(y) = V_i \circ \Phi(y)$, $i = 1, \ldots, n-1$, are the functionally independent first integrals of $G(y)$. Since V_{iG} is also the first integral of By, it means that By and $G(y)$ are orthogonal to the $(n-1)$-dimensional linear space formed by the gradient of V_{iG} for $i = 1, \ldots, n-1$, at any point in a suitable neighborhood of the origin. So, $G(y)$ is parallel to By. Consequently, we have $G(y) = \operatorname{diag}(\mu_1 y_1, \ldots, \mu_n y_n)(1 + h(y))$, where $h(y)$ is a series starting from terms of degree at least 1, and its monomials, denote $h_m y^m$, satisfying $\mu^m = 1$.

The diffeomorphism $F(x)$ has $n-1$ functionally independent first integrals, so there exist $n-1$ linearly independent vectors $m_i = (m_{i1}, \ldots, m_{in}) \in \mathbb{Z}_+^n$ such that $\mu^{m_i} = 1$ for $i = 1, \ldots, n-1$. From this we can prove that there exits $\sigma > 0$ such that if $\mu^m - \mu_i \neq 0$ for $m \in \mathbb{Z}_+^n$ and $|m| \geqslant 2$, we have $|\mu^m - \mu_i| \geqslant \sigma$.

Working in a similar way to the proof of Lemma 2.6, we can prove that $G(y)$ and $\Phi(y)$ are convergent in a suitable neighborhood of the origin. This implies that the diffeomorphism $F(x)$ is analytically equivalent to its distinguished normal form $G(y)$ via the analytic transformation $x = \Phi(y)$. We finish the proof of the theorem. $\square$

Proof of Theorem 1.3. Let $V_1(x), \ldots, V_{n-1}(x)$ be the $n-1$ functionally independent analytic first integrals of $F(x)$. Then each level surface $V_i(x) = c_i$ is invariant under the action of $F(x)$. So, each orbit of $F(x)$ is contained in $\bigcap_{i=1}^{n-1}\{x \in \mathcal{M};\ V_i(x) = c_i\} := \gamma_c$ for some $c = (c_1, \ldots, c_{n-1}) \in \mathbb{R}^{n-1}$.

Set

$$\mathcal{X}(x) = \nabla V_1(x) \times \cdots \times \nabla V_{n-1}(x), \quad x \in \mathcal{M},$$

where ∇ denotes the gradient of a differentiable function, and $\times$ the cross product of vectors in $\mathbb{R}^n$. In $\mathbb{R}^n$, the cross product of $n-1$ vectors $v_1, \ldots, v_{n-1}$ is again a vector, and is defined as

$$(v_1 \times \cdots \times v_{n-1}) \cdot w = \det \begin{pmatrix} w \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix},$$

for arbitrary $w \in \mathbb{R}^n$, where the dot denotes the inner product of two vectors in $\mathbb{R}^n$. Clearly, $v = v_1 \times \cdots \times v_{n-1}$ is orthogonal to each v_i for $i = 1, \ldots, n-1$.

From the definition of γ_c and $\mathcal{X}(x)$, we know that $\mathcal{X}(x)$ is an analytic vector field and is tangent to each γ_c at $x \in \gamma_c$. So, γ_c is the orbit of $\mathcal{X}(x)$. This proves that the set of orbits of $\mathcal{X}(x)$ is formed by $\{\gamma_c;\ c \in \mathbb{R}^n\}$, and that any orbit of $F(x)$ is contained in an orbit of $\mathcal{X}$.

We claim that $\mathcal{X}$ is an embedding vector field of $F(x)$. The idea for proving this claim follows from [5]. Firstly, we have

$$\det\big(DF(x)\big)\mathcal{X}(x) = DF(x)\big(\big(DF(X)\big)^t V_1(x) \times \cdots \times \big(DF(x)\big)^t V_{n-1}(x)\big). \tag{3.2}$$

Because for any $w(x) \in T_x\mathcal{M}$, the tangent space of $\mathcal{M}$ at x,

$$\begin{aligned} & w(x) \cdot DF(x)\big(\big(DF(X)\big)^t V_1(x) \times \cdots \times \big(DF(x)\big)^t V_{n-1}(x)\big) \\ &\quad = \big(DF(x)\big)^t w(x) \cdot \big(\big(DF(X)\big)^t V_1(x) \times \cdots \times \big(DF(x)\big)^t V_{n-1}(x)\big) \\ &\quad = \det \begin{pmatrix} (DF(x))^t w(x) \\ (DF(x))^t V_1(x) \\ \vdots \\ (DF(x))^t V_{n-1}(x) \end{pmatrix} = \det\big(DF(x)\big) \det \begin{pmatrix} w(x) \\ V_1(x) \\ \vdots \\ V_{n-1}(x) \end{pmatrix}. \end{aligned}$$

Secondly, the following hold

$$\big(DF(x)\big)\mathcal{X}(x) = \big(\det DF(x)\big)\mathcal{X}\big(F(x)\big). \tag{3.3}$$

Indeed, since $V_i(F(x)) = V_i(x)$ for $i = 1, \ldots, n-1$, we have

$$\big(DF(x)\big)^t \nabla V_i\big(F(x)\big) = \nabla V_i(x). \tag{3.4}$$

Then we get

$$\begin{aligned} \big(DF(x)\big)\mathcal{X}(x) &= \big(DF(x)\big)\big(\big(DF(X)\big)^t V_1\big(F(x)\big) \times \cdots \times \big(DF(x)\big)^t V_{n-1}\big(F(x)\big)\big) \quad \text{by (3.4)} \\ &= \det\big(DF(x)\big)\mathcal{X} \circ F(x) \quad \text{by (3.2)}. \end{aligned}$$

By the assumption that the diffeomorphism is volume-preserving, i.e. $\det(D(F(x))) = 1$, we obtain from (3.3) that $\mathcal{X}$ is an analytic embedding vector field of $F(x)$. This completes the proof of the theorem. $\square$

Remark 3. In Theorem 1.3 we assume that the integrable diffeomorphism is volume-preserving. We do not know if a general analytic integrable diffeomorphism can also be embedded in an analytic or a C^∞ flow. The possible solution to this problem is to find a smooth function $\rho(x)$ such that $\rho(x)\mathcal{X}(x)$ is an embedding vector field. But it follows from Theorem 12 of [5] that the existence of such a ρ is equivalent to the existence of solutions of the functional equation $\rho(F(x)) = \det(DF(x))\rho(x)$. We have no idea if this last equation has a solution provided that $F(x)$ is not volume-preserving.

Remark 4. Theorem 1.3 is also correct if we replace the analyticity by C^k smoothness for $k = 1, \ldots, \infty$. But the embedding flow is C^{k-1} for $k \neq \infty$.

Remark 5. It is an open problem *if any C^k, $k = 1, \ldots, \infty, \omega$, smooth integrable diffeomorphism defined on a C^k smooth manifold has a C^{k-1} smooth embedding flow on the manifold*, where we use the convention: $\infty - 1 = \infty$ and $\omega - 1 = \omega$.

References

[1] Yu.N. Bibikov, Local Theory of Nonlinear Analytic Ordinary Differential Equations, Lecture Notes in Math., vol. 702, Springer-Verlag, Berlin, 1979.
[2] J. Chavarriga, H. Giacomini, J. Giné, J. Llibre, On the integrability of two-dimensional flows, J. Differential Equations 157 (1999) 163–182.
[3] J. Chavarriga, H. Giacomini, J. Giné, J. Llibre, Local analytic integrability for nilpotent centers, Ergodic Theory Dynam. Systems 23 (2003) 417–428.
[4] J. Chen, Y. Yi, X. Zhang, First integrals and normal forms for germs of analytic vector fields, preprint.
[5] A. Cima, A. Gasull, V. Mañosa, Studying discrete dynamical systems through differential equations, J. Differential Equations 244 (2008) 630–648.
[6] H. Giacomini, J. Gine, J. Llibre, The problem of distinguishing between a center and a focus for nilpotent and degenerate analytic systems, J. Differential Equations 227 (2006) 406–426.
[7] E. Hille, Ordinary Differential Equations in the Complex Domain, Wiley, New York, 1976.
[8] H. Ito, Convergence of Birkhoff normal forms for integrable systems, Comment. Math. Helv. 64 (1989) 412–461.
[9] H. Ito, Integrability of Hamiltonian systems and Birkhoff normal forms in the simple resonance case, Math. Ann. 292 (1992) 411–444.
[10] Weigu Li, J. Llibre, Xiang Zhang, Extension of Floquet's theory to nonlinear periodic differential systems and embedding diffeomorphisms in differential flows, Amer. J. Math. 124 (2002) 107–127.
[11] J. Moser, The analytic invariants of an area-preserving mapping near a hyperbolic fixed point, Comm. Pure Appl. Math. 9 (1956) 673–692.
[12] J. Palis, Vector fields generate few diffeomorphisms, Bull. Amer. Math. Soc. 80 (1974) 503–505.
[13] H. Poincaré, Sur l'intégration des équations différentielles du premier order et du premier degré, I and II, Rend. Circ. Mat. Palermo 5 (1891) 161–191; Rend. Circ. Mat. Palermo 11 (1897) 193–239.
[14] C.L. Siegel, On the integrals of canonical systems, Ann. of Math. 42 (1941) 806–822.
[15] S. Sterberg, Local contractions and a theorem of Poincaré, Amer. J. Math. 79 (1957) 809–824.
[16] Xiang Zhang, Embedding smooth homeomorphisms in the flows, preprint, 2007.
[17] N.T. Zung, Convergence versus integrability in Birkhoff normal form, Ann. of Math. 161 (2005) 141–156.